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The method of integrodifferential relations for analysing the natural oscillations of membranes☆

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ABSTRACT

An approach to solving of problems of the free oscillations of membranes, based on the method of integrodifferential relations, is proposed. The variational properties of the integrodifferential problems are investigated and their relation to classical variational principles is demonstrated. Integral and local criteria for the quality of the approximate solutions are proposed. A numerical-analytical algorithm is developed for finding the characteristic frequencies and modes of oscillations. Problems involving the free transverse motions of circular and elliptic membranes are considered as an example.

Despite of the fact that the first papers dealing with the natural oscillations of a membrane date back to the nineteenth century when analytical solutions were obtained for a number of simple domains,[1,2] questions concerning the construction of approximate solutions and estimates of the quality for membranes of complex shape, structurally inhomogeneous membranes, etc. remain open. Different methods, based on variational principles of mechanics, are used for such problems.[3] The oscillations of triangular membranes have been investigated using a variational approach and piecewise-polynomial approximations.[4] We mention the method of accelerated convergence,[5,6] which has been used to find the frequencies of modes of oscillation of elliptic membranes with different boundary conditions.

In cases when it is difficult to obtain an exact solution, approximate approaches are used, based, for example, on the finite element method which reduces the initial variational formulation to a finite-dimensional system of equations.[3] Note that the schemes developed for this method have a number of limitations. For instance, when constructing a finite-dimensional approximation of the deformed state of a membrane it is quite difficult to analyse the natural modes of oscillation and to obtain a bilateral estimate of the convergence of the numerical process. This does not permit effective control of the errors in the calculations, associated, for example, with the discretization of the problem, rounding off, numerical integration, etc.

The essence of the method of integrodifferential relations, which is developed here, lies in the fact that certain governing relations in a local form are represented in integral form. At the same time, force and geometric variables are introduced into the treatment and, moreover, these variables are independent.[7–10]

The integrodifferential formulations of the problem of the free oscillations of a membrane, proposed below, are primarily directed to identifying the governing relations and associations with the formulations of this problem in the local and integral forms (classical variational principles) and to developing corresponding approaches and methods of solution. Note that the method of integrodifferential relations also turns out to be useful in the treatment of other boundary-value problems of mathematical physics.[11–13]

1. Formulation of the boundary value problem

A theoretical membrane is a flexible and infinitesimally thin platelet which is stretched in all directions by a force which is so great that it does not change in a perceptible manner during the course of the oscillations being considered. It is assumed that the potential energy of the membrane is proportional to the change in the area of the surface, and the coefficient of proportionality τ is called the tension.[1]

Consider an elastic membrane occupying a certain plane domain Ω with a boundary γ. We will denote a displacement of a point of the membrane perpendicular to the (x,y) plane by $u(x,y)$ and we will assume this displacement to be sufficiently small in the sese that higher powers of the quantities u, du/dx, du/dy can be neglected compared with the lower powers. The expression for the area of the surface can

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be written in the form of the integral

$$\iint_{\Omega}\left(1+\frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]\right)d\Omega$$

and, as the required expression for the potential energy, when account is taken of the coefficient of proportionality τ, we obtain

$$W = \frac{\tau}{2}\iint_{\Omega}\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right)d\Omega \tag{1.1}$$

Taking account of the fact that the elastic forces arising in the membrane can only be compensated by inertial forces, after variation of the functional (1.1) and integration by parts we obtain Euler's equation for this functional

$$\tau(\partial^2 u/\partial x^2+\partial^2 u/\partial y^2)-\rho\partial^2 u/\partial t^2 = 0 \tag{1.2}$$

Here, ρ is the surface density of the membrane. The constitutive equation, which relates the force variables (the internal stresses) and the geometric variables (the deformations of the membrane) are not explicitly taken into account in deriving the equations of motion of the membrane. In order to use the ideas of the method of integrodifferential relations, we introduce into the treatment a stress vector function $\mathbf{s}$ and a momentum density function p which are connected to the displacement function u by the relations

$$s_x = \tau\partial u/\partial x; \quad s_y = \tau\partial u/\partial y, \quad p = \rho\partial u/\partial t \tag{1.3}$$

The physical meaning of the stress vector lies in the fact that S_x and S_y are the projections of the stresses τ onto the z axis. By analogy with the theory of elasticity,[9] we set up the strain vector

$$\boldsymbol{\varepsilon}^0 = (\partial u/\partial x, \partial u/\partial y)^T = \nabla u \tag{1.4}$$

The first two equations of (1.3) can then be represented in vector form

$$\mathbf{q} \equiv \boldsymbol{\varepsilon}^0 - \tau^{-1}\mathbf{s} = 0 \tag{1.5}$$

Here, τ is the analogue of an elastic moduli matrix and $\mathbf{q}$ is a dimensionless vector. The relations linking the momentum density function and the velocity of the points of the membrane is written in the form

$$\upsilon \equiv \partial u/\partial t - \rho^{-1}p = 0 \tag{1.6}$$

The auxiliary function υ, which has the dimension of velocity, is introduced for notational convenience.

Using relations (1.3), the equation of motion (1.2) can be expressed in terms of the stress vector and the momentum density function

$$\nabla\cdot\mathbf{s} - \partial p/\partial t = 0 \tag{1.7}$$

Since expression (1.7) connects elastic and inertial bulk forces, we shall henceforth call this relation the dynamic equilibrium equation. It is obvious that Eqs. (1.4) - (1.7) are equivalent to the equation of motion (1.2).

We will confine ourselves to two cases (the Dirichlet and Neumann conditions[2]) when the boundary conditions are written in the displacements: $u = 0$ or in the stresses:

$$\mathbf{s}\cdot\mathbf{n} = 0, \{x, y\} \in \gamma \tag{1.8}$$

where n is the unit vector normal to the boundary γ.

Different variational method have been widely used to find approximate solutions of problems of the free oscillations of membranes. In the case when the strain energy density and kinetic energy functions

$$A_u = \frac{1}{2}\tau(\boldsymbol{\varepsilon}^0\cdot\boldsymbol{\varepsilon}^0), \quad K_u = \frac{1}{2}\rho\left(\frac{\partial u}{\partial t}\right)^2 \tag{1.9}$$

exist and boundary conditions (1.8) do not change when the displacements very, Hamiltons principle[3]

$$\delta H = \delta\int_0^{t_f}[T-\Pi]dt = 0, \quad T = \int_{\Omega}K_u d\Omega, \quad \Pi = W = \int_{\Omega}A_u d\Omega \tag{1.10}$$

follows from the virtual work principle in the case of periodic problems.

Note that, in the case of this formulation, the displacement fields u must strictly satisfy kinematic relations (1.4), constitutive equations (1.5) and (1.6) and boundary conditions (1.8) in displacements. The boundary conditions in stresses are implicitly taken into account in the functional H from (1.10) and are satisfied in an integral sense. Strict satisfaction of equilibrium equation (1.7) is also not required.

If a function for the additional energy density, expressed in stresses and an additional kinetic energy density function

$$A_s = \frac{1}{2}\tau^{-1}(\mathbf{s}\cdot\mathbf{s}), \quad K_p = \frac{1}{2}\rho^{-1}p^2 \tag{1.11}$$

exist and it is assumed that boundary conditions (1.8) do not change when the stresses are varied, the principle of the stationarity of the functional H_c which is additional to the Hamiltonian H follows from the principle of additional virtual work

$$\delta H_c = \delta\int_0^{t_f}[T_c - \Pi_c]dt = 0, \quad T_c = \int_\Omega K_p d\Omega, \quad \Pi_c = \int_\Omega A_s d\Omega \tag{1.12}$$

In the case of this variational principle, the stress field s and the momentum density field p must strictly satisfy equilibrium equation (1.7), constitutive equations (1.5) and (1.6) and boundary conditions (1.8) in stresses. At the same time, the boundary conditions in displacements are taken into account in the functional H_c from (1.12) and strict satisfaction of kinematic conditions (1.4) is not required.

It is assumed (see Ref. 3) that, within the limits of the assumptions of linear theory, these two approaches to the formulation of variational principles are mutual and equivalent to one another. Note that the approximate displacement fields u obtained from Hamilton's principle enable us to recover the corresponding stress fields $\mathbf{s}^0$ using the linear kinematic relations (1.4), constitutive equations (1.5) and (1.6), and

$$\mathbf{s}^0 \equiv \tau\boldsymbol{\varepsilon}^0(u) \tag{1.13}$$

On the other hand, the approximate stress fields $\mathbf{s}$, found from the additional principle (1.12), can be related to the corresponding strain fields ε

$$\boldsymbol{\varepsilon} \equiv \tau^{-1}\mathbf{s} \tag{1.14}$$

Here, it is quite difficult in the general case to recover the displacement fields u using the strains ε which have been found since, in order to do this, it is necessary to solve the overdetermined system of differential equations (1.4). Note that the stress fields $\mathbf{s}$ and $\mathbf{s}^0$, obtained by these two approaches cannot be identical $\mathbf{s} \neq \mathbf{s}^0$ in the general case.

We also point out that classical variational principles are obtained from the assumption that positive definite forms exist when given constitutive laws are strictly satisfied.

The equilibrium equation in local form (1.7) follows from the existence of a stress vector and its definite smoothness (the existence of first derivatives of the stress functions). These relations are a reflection of Newton's third law and, generally speaking, must be strictly satisfied.

Kinematic equations (1.4), with assumptions on the smallness of the deformations and displacements, are relations describing the geometry of the deformed membrane. In the linear theory, these relations are also assumed to be strictly satisfied.

Specification of the boundary conditions is dictated by the type of problem being solved. The form of the boundary conditions must be determined before solving the problem.

The requirement that the constitutive laws are strictly satisfied is unnecessarily rigid. Generally speaking, the introduction into the treatment of a linear constraint between the stresses and strains is primarily due to a desire to simplify the problem. Real membranes satisfy such a linear dependence approximately.

2. The method of integrodifferential relations

In order to formulate the integrodifferential problem, we will rewrite relations (1.5) and (1.6) in the form of the integral

$$\Phi = \int_0^{t_f}\int_\Omega \varphi(u, p, \mathbf{s})d\Omega dt = 0; \quad \varphi = \frac{1}{2}(\tau\mathbf{q}\cdot\mathbf{q} + \alpha\rho\upsilon^2), \quad \mathbf{q} = \nabla u - \frac{\mathbf{s}}{\tau}, \quad \upsilon = \frac{\partial u}{\partial t} - \frac{p}{\rho} \tag{2.1}$$

Here, α is an arbitrary real number. It is obvious that the functional Φ is non-negative for any functions u, p and s when $\alpha \geq 0$. According to equality (2.1), the local conditions (1.5) and (1.6) must be satisfied everywhere within the domain Ω with the exception, perhaps, of points constituting a set of zero measure. Using this integral relation, the boundary value problem (1.4) - (1.8) can be reduced to the following variational problem[12]: it is required to find functions u^*, p^*, $\mathbf{s}^*$ which strictly satisfy equilibrium equation (1.7), boundary conditions (1.8) and give stationary values to the functional Φ:

$$\delta\Phi(u, p, \mathbf{s}) = 0 \tag{2.2}$$

In the case when problem (1.3) - (1.8) has a solution u^*, p^*, $\mathbf{s}^*$, the value of the functional Φ is equal to zero:

$$\Phi(u^*, p^*, \mathbf{s}^*) \equiv 0 \tag{2.3}$$

With this a representation of the constitutive laws of it is assumed that the stress vector $\mathbf{s}$, the momentum density p and the displacements u are independent and the state of a membrane is described by two types of variables: force variables and geometric variables. The functions s_x, and s_y, constituting the stress vector $\mathbf{s}$, and the function p are classified as force variables and the function u is classified as a geometric variable.

In order to find the natural motions of a membrane, we will seek a solution of integrodifferential problem (2.2), (1.7), (1.8) in the form

$$u = \tilde{u}e^{i\omega t}, \quad p = i\tilde{p}e^{i\omega t}, \quad \mathbf{s} = \tilde{\mathbf{s}}e^{i\omega t} \tag{2.4}$$

Real functions of Cartesian coordinates are denoted by a tilde, which is subsequently omitted.

Since, when account is taken of relations (2.4), problem (1.4) - (1.8) is not explicitly time-dependent, the functional Φ and the equilibrium equations for finding the natural frequencies ω and the natural modes of oscillation take the form

$$\Phi = \int_\Omega \varphi d\Omega dt = 0; \quad \varphi = \frac{1}{2}(\tau \mathbf{q}\cdot\mathbf{q} + \alpha\rho v^2), \quad \mathbf{q} = \nabla u - \frac{\mathbf{s}}{\tau}, \quad v = \omega u - \frac{p}{\rho} \tag{2.5}$$

$$\nabla\cdot\mathbf{s} + \omega p = 0 \tag{2.6}$$

When investigating the properties of integrodifferential problem (2.5), (2.6) with boundary conditions (1.8), we will consider the two cases when $\alpha \neq 0$ and $\alpha = 0$.

If $\alpha \neq 0$, then we assume that the real and arbitrarily chosen displacements, momentum densities and stresses are denoted by u^*, p^*, $\mathbf{s}^*$ and u, p and $\mathbf{s}$ respectively and put

$$u = u^* + \delta u, \quad p = p^* + \delta p, \quad \mathbf{s} = \mathbf{s}^* + \delta\mathbf{s}$$

Then,

$$\Phi(u, p, \mathbf{s}) = \delta_u\Phi + \delta_p\Phi + \delta_{\mathbf{s}}\Phi + \delta^2\Phi$$

where $\delta_u\Phi$, $\delta_p\Phi$ and $\delta_s\Phi$ are first variations with respect to u, p and $\mathbf{s}$, and $\delta^2\Phi$ is the second variation of the functional Φ. After integrating by parts and taking account of the relation between the variations δp and δs

$$\delta p = -\omega^{-1}\nabla\cdot\delta\mathbf{s} \tag{2.7}$$

which follows from the equation for the dynamic equilibrium of the membrane (2.6), the first variations, lines in δu, δp and δs, can be written as follows:

$$\delta_u\Phi = \int_\Omega [\alpha\omega\rho v - \tau\nabla\cdot\mathbf{q}]\delta u d\Omega + \tau\int_\gamma (\mathbf{n}\cdot\mathbf{q})\delta u d\gamma$$

$$\delta_p\Phi + \delta_{\mathbf{s}}\Phi = -\int_\Omega [\alpha\omega^{-1}\nabla v + \mathbf{q}]\delta\mathbf{s} d\Omega + \alpha\omega^{-1}\int_\gamma v(\mathbf{n}\cdot\delta\mathbf{s}) d\gamma \tag{2.8}$$

The conditions under which the first variations of the functional Φ vanish (the stationarity conditions) have the form

$$\tau\Delta v + \omega^2\rho v = 0, \quad \omega\mathbf{q} + \alpha\nabla v = 0$$
$$v|_{\{x,y\}\in\gamma} = 0 \text{ или } \nabla v\cdot\mathbf{n}|_{\{x,y\}\in\gamma} = 0 \tag{2.9}$$

It follows from (2.9) that the local constitutive equations (1.5) and (1.6) are conditions for the stationarity of the functional Φ, which, together with onstraints (1.7) - (1.8), constitute a complete system of equations describing the motion of the membrane.

The second variation

$$\delta^2\Phi = \int_\Omega \varphi(\delta u, \delta p, \delta\mathbf{s}) d\Omega$$

is quadratic in δu, δp and $\delta\mathbf{s}$, non-negative when $\alpha > 0$ as a consequence of the non-negativity of the integrand and non - sign -definite when $\alpha < 0$.

When $\alpha = -1$, the first variations of the functional Φ can be represented in the form

$$\delta_u\Phi = \int_\Omega [\tau\Delta u + \omega^2\rho u]\delta u d\Omega + \int_\gamma \mathbf{n}\cdot(\tau\nabla v - \mathbf{s})\delta u d\gamma$$

$$\delta_p\Phi + \delta_{\mathbf{s}}\Phi = \int_\Omega [\omega^{-2}\rho^{-1}\nabla(\nabla\cdot\mathbf{s}) + \tau^{-1}\mathbf{s}]\delta\mathbf{s} d\Omega + \int_\gamma (\omega u + \omega^{-1}\rho^{-1}\nabla\cdot\mathbf{s})(\mathbf{n}\cdot\delta\mathbf{s}) d\gamma \tag{2.10}$$

These relations are obtained by corresponding integration by parts taking account of variational constraint (2.7). In this case, the first variations $\delta_u\Phi$ depend solely on the displacements and the variations $\delta_p\Phi + \delta_{\mathbf{s}}\Phi$ depend solely on the stresses. In this case, the problem of finding the stationary values of the functional Φ splits into two independent classical variational problems: to find the functions u from the conditions for the Hamiltonian (1.10) to the stationary and to find the stress field $\mathbf{s}$ and the momentum field p from the principle of stationarity of the additional Hamiltonian functional (1.12).

In the case when $\alpha = 0$, integrodifferential problem (2.5) can be reformulated in the form

$$\Phi_0 = \Phi|_{\alpha=0} = \int_\Omega \tau\mathbf{q}\cdot\mathbf{q} d\Omega = 0, \quad \mathbf{q} = \nabla u - \tau^{-1}\mathbf{s}, \quad \nabla\cdot\mathbf{s} + \omega^2\rho u = 0$$
$$u|_{\{x,y\}\in\gamma} = 0 \text{ или } \mathbf{s}\cdot\mathbf{n}|_{\{x,y\}\in\gamma} = 0 \tag{2.11}$$

It is assumed here that relation (1.6), linking the momentum density function p and the velocity function $\delta u/\delta t$, is strictly satisfied. As a consequence of the positive definiteness of the functional Φ_0, problem (2.11) can be reduced to the problem of finding the absolute minimum of this functional with respect to the displacements u and the stresses $\mathbf{s}$. In this case, the variations Φ_0 have the form

$$\delta\Phi_0 = \delta_u\Phi_0 + \delta_s\Phi_0 = 0$$

$$\delta_u\Phi_0 = -\tau\int_\Omega \nabla\cdot\mathbf{q}\,\delta u\,d\Omega + \tau\int_\gamma (\mathbf{n}\cdot\mathbf{q})\delta u\,d\gamma, \quad \delta_s\Phi_0 = -\int_\Omega \mathbf{q}\cdot\delta\mathbf{s}\,d\Omega$$

$$\delta u = -\omega^2\rho^{-1}\nabla\cdot\delta\mathbf{s}$$

$$\delta\Phi_0 = -\int_\Omega [\tau\omega^{-2}\rho^{-1}\nabla(\nabla\cdot\mathbf{q}) + \mathbf{q}]\cdot\delta\mathbf{s}\,d\Omega + \tau\omega^{-2}\rho^{-1}\int_\gamma (\nabla\cdot\mathbf{q})(\mathbf{n}\cdot\delta\mathbf{s})\,d\gamma + \tau\int_\gamma (\mathbf{n}\cdot\mathbf{q})\delta u\,d\gamma \tag{2.12}$$

3. An algorithm for the numerical solution of the problem

We will consider one of the possible algorithms for the conditional minimization of the functional Φ_0 in problem (2.11) and use Ritz's method, described, for example, in Ref. 3. For this purpose, we represent the required functions u, s_x, s_y in the form of the finite-dimensional polynomial expansions

$$u = \sum_{k=0}^{n_u}\sum_{l=0}^{k} u^{kl}x^l y^{k-l}, \quad s_x = \sum_{k=0}^{n_s}\sum_{l=0}^{k} s_x^{kl}x^l y^{k-l}, \quad s_y = \sum_{k=0}^{n_s}\sum_{l=0}^{k} s_y^{kl}x^l y^{k-l} \tag{3.1}$$

where n_u and n_s are the degrees of the complete polynomials.

After substituting approximations (3.1) into the equilibrium equations and the boundary conditions, the satisfaction of these relations at each point (x, y) is equivalent to the coefficients (which are linear combinations of u^{ki}, s_x^{kl} and s_y^{kl}) of each monomial in the corresponding polynomials being equal to zero. Gathering the remaining undetermined constants u^{ki}, s_x^{kl} and s_y^{kl} into a N_u-dimensional scheme parameter vector $\mathbf{u}$ and integrating the first relation of (2.5), we represent the function Φ_0 as the function of N_u variables

$$\Phi_0 = \Phi_0(\mathbf{u}) = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} \tag{3.2}$$

The vector $\mathbf{u}$ is found from the linear algebraic system of equations

$$\frac{\partial\Phi_0}{\partial u_1} = 0, \quad \frac{\partial\Phi_0}{\partial u_2} = 0, \ldots, \frac{\partial\Phi_0}{\partial u_{N_u}} = 0 \tag{3.3}$$

System of equations (3.3) can be rewritten in the matrix form

$$\mathbf{K}\mathbf{u} = \mathbf{0} \tag{3.4}$$

Unlike the classical variational approach, based on the principle of the stationarity of the Hamiltonian (1.10), in which the eigenvalues $\lambda = \rho/\tau^{-1}\omega^2$ occur linearly in the coefficients of the matrix $\mathbf{K}$, in the method of integrodifferential relations the coefficients of $\mathbf{K}$ are quadratically related to the parameter λ. The matrix $\mathbf{K}$ has the following structure

$$\mathbf{K}(\lambda) = \lambda^2\mathbf{A} + \lambda\mathbf{B} + \mathbf{C}, \quad \mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^{N_u\times N_u} \tag{3.5}$$

The eigenvalues λ for Eq. (3.4) can be found from the condition that the determinant

$$\det\mathbf{K}(\lambda) = 0 \tag{3.6}$$

is equal to zero.

The eigenvectors $\mathbf{u}_\lambda$ are found from system of equations (3.4). Here, the value of the quadratic functional (3.2) is precisely equal to zero.

Taking account of the fact that, with respect to the parameter λ, the determinant of the matrix $\mathbf{K}$ is a polynomial with real coefficients, it follows from a basic theorem of algebra that the set of solutions of Eq. (3.6) consists of the real roots and pairs of complex conjugate roots of this polynomial. If the values of the frequency ω which are found are real, then they, together with the eigenvectors $\mathbf{u}_\lambda$, strictly satisfy all the relations of integrodifferential problem (2.11). These ω and $\mathbf{u}_\lambda$ are an exact solution of the problem of the natural oscillations of a membrane.

If the finite-dimensional approximations (3.1) used do not enable us to find exact solutions, then the frequencies ω, satisfying Eq. (3.6), can only be complex-valued. The components of the eigenvectors $\mathbf{u}_\lambda$ also turn out to be complex. We will explain this situation by means of an example. It is well known that the analytical solution of the problem of the free oscillations of a thin circular membrane is described in terms of trigonometric and Bessel functions. At the same time it is clear that the finite-dimensional polynomial functions from expansions (3.1) can only roughly approximate these solutions.

On the one hand, complex values of ω contradict the assumptions made in (2.4). However, on the other hand, the real parts $\tilde{\omega} = Re(\omega)$ and $\tilde{\mathbf{u}}_\lambda = Re(\mathbf{u}_\lambda)$ can serve as an approximate solution of the problem of the oscillations of a membrane. The following criterion can be used

Table 1

Models of oscillations	λ^*	$\mathrm{Re}\lambda$-λ^*	$\mathrm{Im}\lambda$	Δ, %
I	5.783	5.8×10^{-20}	3.1×10^{-11}	7.9×10^{-21}
II, III	14.682	2.2×10^{-19}	1.1×10^{-8}	5.7×10^{-17}
I, IV	26.375	3.9×10^{-13}	1.3×10^{-5}	2.5×10^{-11}
I	30.471	4.0×10^{-12}	4.2×10^{-5}	1.9×10^{-10}
II, III	40.706	1.9×10^{-11}	9.5×10^{-5}	1.0×10^{-9}
II,III	49.218	4.4×10^{-10}	4.5×10^{-4}	8.5×10^{-9}
I, IV	57.583	1.4×10^{-7}	7.3×10^{-3}	1.9×10^{-6}
I, IV	70.850	3.1×10^{-6}	3.3×10^{-2}	2.2×10^{-5}
I	74.887	7.1×10^{-6}	4.9×10^{-2}	4.3×10^{-5}
II, III	76.939	9.2×10^{-7}	1.9×10^{-2}	6.4×10^{-6}

to estimate the quality of the approximate solutions obtained

$$\Delta_\lambda = \frac{\Phi_0(\tilde{\mathbf{u}}_\lambda)}{W(\tilde{\mathbf{u}}_\lambda)} < \mu \tag{3.7}$$

where $W(\tilde{\mathbf{u}}_\lambda)$ is the potential energy of the membrane (1.1), calculated from the approximate solution $\tilde{\mathbf{u}}_\lambda$ and μ is a given positive number, characterizing the accuracy of the solution.

4. Oscillations of circular and elliptic membranes

We will now demonstrate the proposed algorithm by finding the natural oscillations of circular and elliptic membranes as an example.

Example 1. Consider a circular membrane of radius $R = 1$, fixed along its external contour. We specify that the stress in the membrane τ and the surface density ρ are equal to unity.

In accordance with the proposed algorithm, we specify the unknown functions u, s_x, s_y in the form of finite-dimensional polynomial expansions (3.1) and require that the powers of the two-dimensional polynomials (3.1) should be connected by the relation

$$n_s = n_u + 1 \tag{4.1}$$

There are therefore $N_T = (3n_u + 7)(n_u + 2)/2$ free parameters available for finding the approximate eigenvalues and characteristic modes.

Satisfying the equilibrium equations is equivalent to the coefficients of each monomial in the polynomial, which is obtained after substituting expansions (3.1) into Eq. (2.6), being equal to zero. The number of permitted relations is given by the following formula $N_r = (3n_u + 7)(n_u + 2)/2$.

In order to satisfy the Dirichlet boundary condition in (1.8), it is more convenient to change to the polar coordinates

$$x = r\cos\varphi, \quad y = r\sin\varphi \tag{4.2}$$

We then obtain a polynomial in, for example, the functions $\sin\varphi$ and $\cos^\alpha\varphi (0 \le \alpha \le n_u)$. The strict satisfaction of the boundary condition guarantees that the coefficients of each monomial are equal to zero, and it is necessary to satisfy $N_b = 2n_u + 1$ relations. Similarly, it is possible to satisfy the Neumann boundary condition in the stresses.

There are therefore $N_u = n_u^2 + 3n_u + 5$ free parameters available for the solution of the variational problem (2.11).

In order to increase the efficiency of the algorithm, we will use the symmetry properties of the boundary value-problem being considered. It can be shown that, in the case of a membrane with a boundary γ, which is symmetrical about the axes of the Cartesian system of coordinates, the initial problem splits into four independent subproblems in the following manner. The first problem describes the oscillations of a membrane (type I) for which the displacements function u is symmetrical about to the coordinate axes ($u(x,y) = u(-x,y) = u(x,-y)$, The second type of oscillations (II) specifies displacements which are symmetrical about to the x-axis and antisymmetric with respect to the other axis ($u(x,y) = u(-x,y) = -u(x,-y)$. For the third subproblem (type III), $-u(x,y) = -u(-x,y) = u(x,-y)$. The antisymmetry of the displacement fields with respect to the coordinate axes ($u(x,y) = -u(-x,y) = -u(x,-y)$) characterizes the last type of oscillations (IV).*****

The polynomial representations (3.1) are found to be convenient for the approximating displacement and stress fields of the corresponding types of oscillations of the membrane. For example, only the even functions $x^{2n}y^{2n}$ in (3.1) are used to describe symmetric (type I) displacements, and the components s_x and s_y of the stress vector $\mathbf{s}$ are given by the functions $x^{2n+1}y^{2n}$ and $x^{2n}y^{2n+1}$ respectively.

The problem of the natural oscillations of a homogeneous circular membrane clamped along its contour has an analytical solution (see Ref. 1). The first ten eigenvalues λ^* which are squares of the zeroes of the Bessel functions $J_n(\sqrt{\lambda}) = 0$ ($n = 0, 1, 2, \ldots$), and the modes of oscillation corresponding to them are shown in Table 1. We note that multiple frequencies determine two different types of oscillations.

The exact solutions enable us to estimate the effectiveness of the proposed algorithm for finding the approximate natural frequencies and modes of the oscillations. The quantities $\mathrm{Re}\lambda - \lambda^*$ and $\mathrm{Im}\lambda$, that characterize the accuracy with which the eigenvalues are found and, also, the values of the relative integral error Δ according to its definition (3.7),which is a criterion of the quality of the natural modes obtained, are shown in Table 1 for a fixed degree of the polynomials $n_u = 15$. Note the fairly high accuracy to which the lower frequencies are found, which decreases as the magnitude of the eigenvalues λ^* becomes larger. With the given parameter $n_u = 15$, we succeeded in constructing 35 modes of oscillation which satisfy inequality (3.7) for a fixed value of $\mu = 0.01$.

The decrease in the magnitude of the imaginary part of their approximate values (the upper part of the figure) and the decrease (the lower part) in the integral relative error Δ as the approximation parameter n_u is increased are shown in Fig. 1 for the first three eigenvalues mentioned in Table 1 (the solid, dot-dash and dashed lines respectively).

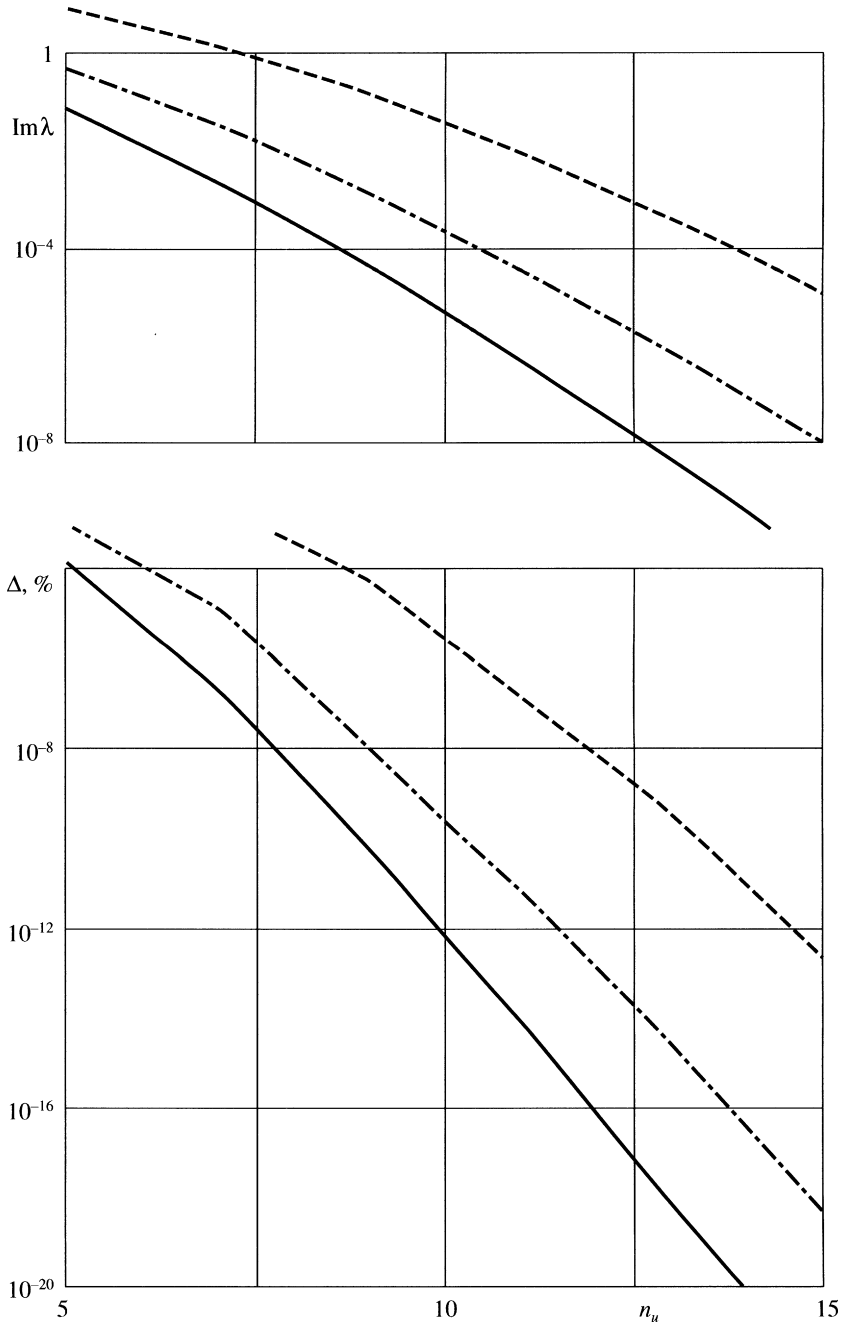

Fig. 1.

Example 2. Consider a homogeneous membrane in the form of an ellipse with semi-axes a and b, is clamped along its contour. As in the preceding example, we specify the tension of the membrane τ and the surface density ρ as being equal to unity and write the equation of the boundary of the membrane in the parametric form

$$x = a\cos\varphi, \quad y = b\sin\varphi \tag{4.3}$$

The following values were chosen in the numerical example: $a = 1$, $b = 0.6$.

The traditional method of solution using the method of separation of the variables reduces the boundary-value problem for the eigenvalues for an elliptic membrane to a system of two parametrically associated ordinary differential equations (Mathieu equations).However, the solution of this system is an extremely difficult problem since there are no sufficiently effective methods for the calculating of Mathieu functions in terms of an argument and a parameter.

The method of accelerated convergence[5] has been used to determine of the natural frequencies and forms of the lower oscillation modes of an elliptic membrane.

Using the polynomial approximations (3.1) of the functions u, s_x, s_y and representation (4.3), as for the case of a circular membrane, it is possible to satisfy the equilibrium equations and the boundary condition (2.11) exactly. As in Example 1, using the symmetry of the ellipse with respect to the coordinate axes, we separate the initial problem into four independent problems.

Table 2

Type of oscillations	Reλ	Imλ
I	10.846	1.4×10^{-9}
III	20.952	9.4×10^{-8}
II	33.938	7.4×10^{-7}
I	35.331	7.1×10^{-5}
IV	48.869	2.0×10^{-4}
III	54.110	6.0×10^{-4}
II	67.523	1.1×10^{-3}
I	70.977	6.7×10^{-4}
I	77.306	4.4×10^{-2}
IV	90.038	5.4×10^{-2}

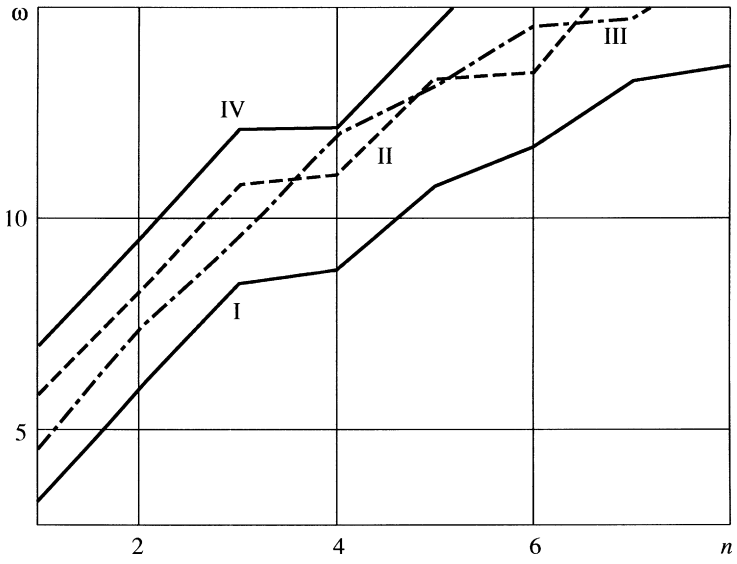

Fig. 2.

The types and approximate values of the first ten eigenvalues $\tilde{\lambda} = Re\lambda$ and the magnitudes of Imλ which characterize the accuracy of the solutions obtained are shown in Table 2. As in the case of a circular membrane, the farly high accuracy to which the lower frequencies are found, that decreases as the values of the eigenvalues $\tilde{\lambda}$ become larger, should be noted. The relative error Δ for an elliptic membrane is higher than for a circular membrane for the same degree of the polynomial approximations n_u. With the given parameter $n_u = 15$, we succeeded in constructing 25 oscillation modes which satisfy inequality (3.7) for the fixed value of $\mu = 0.01$.

The behaviour of the eigenvalues $\tilde{\omega} = \sqrt{\tilde{\lambda}}$, which have been ordered with respect to their magnitude and correspond to different oscillation modes, is shown in Fig. 2. The frequencies $\tilde{\omega}$ can be determined from the graph for integral values of n. It is obvious that, in the case of the specified parameters of the problem, the absolute values of certain frequencies corresponding to different oscillation modes are close. Not taking account of the symmetry properties of the problem can therefore lead to some difficulties in constructing the natural oscillation modes of a membrane.

The method of integrodifferential relations proposed in this paper enables one to analyse the quality of the approximate solutions obtained using local criteria. Consider, for example, the lowest asymmetric mode of oscillation (type IV, Reλ = 48.869 in Table 2). The normalized displacement function for this mode ($u(0, y) = u(x, 0) = 0$) is shown in Fig. 3. The elastic energy density function A_u, defined by the first formula of (1.9), is an important local characteristic of the natural oscillations. Its distribution over the membrane surface for this mode is shown in Fig. 4.

The integrand

$$\varphi_0 = \tau \mathbf{q} \cdot \mathbf{q}, \quad \mathbf{q} = \nabla u - \tau^{-1}\mathbf{s} \tag{4.4}$$

in expression (2.11) for the functional Φ_0 can be chosen as the local criterion determining the error in the approximate values of the function A_u. The non-negative function φ_0 has the dimension of surface energy density. In the case of the chosen polynomial approximation (3.1) ($n_u = 15$), the local energy error, shown in Fig. 5, is close to zero and reaches its maximum values on the membrane boundary.

Example 3. We will now consider a boundary value problem for the eigenvalues (1.4) - (1.7) with the Neumann boundary conditions (1.8) specified on an elliptic boundary which, using the notation of (4.3), can be represented in the form

$$s_x \cos\varphi/a + s_y \sin\varphi/b = 0 \tag{4.5}$$

This problem corresponds to the problems of the free oscillations of a heavy ideal liquid in a vertical elliptic tank of constant depth, and has been considered by a number of authors[6].

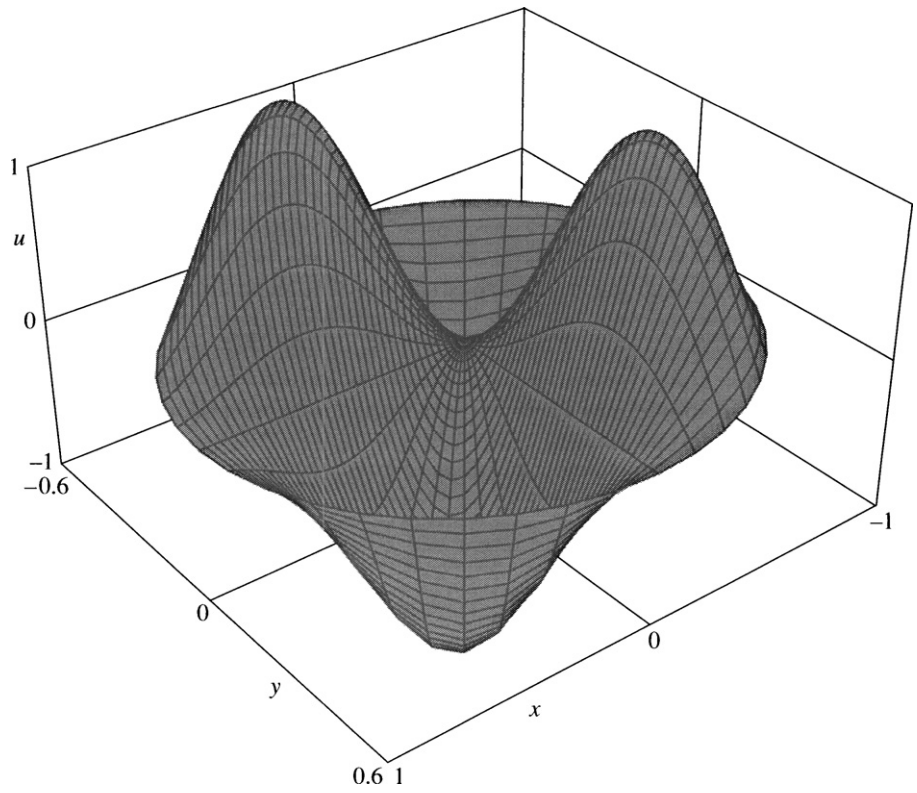

Fig. 3.

We determine the parameters of the problem in the same way as in Example 2 ($\rho = \tau = a = 1$, $b = 0.6$, $n_u = 15$). The types and approximate values of the first ten eigenvalues $\tilde{\lambda} = \mathbf{Re}$ and the magnitudes of Imλ are shown in Table 3. A special feature of the problem with Neumann boundary conditions, as distinct from the Dirichlet problem in Examples 1 and 2, is the fact that the oscillation mode corresponding to the smallest eigenvalue is classified as belonging to the third type of oscillations ($u(x,y) = u(-x,y) = u(x,-y)$). This has been pointed out earlier[6].

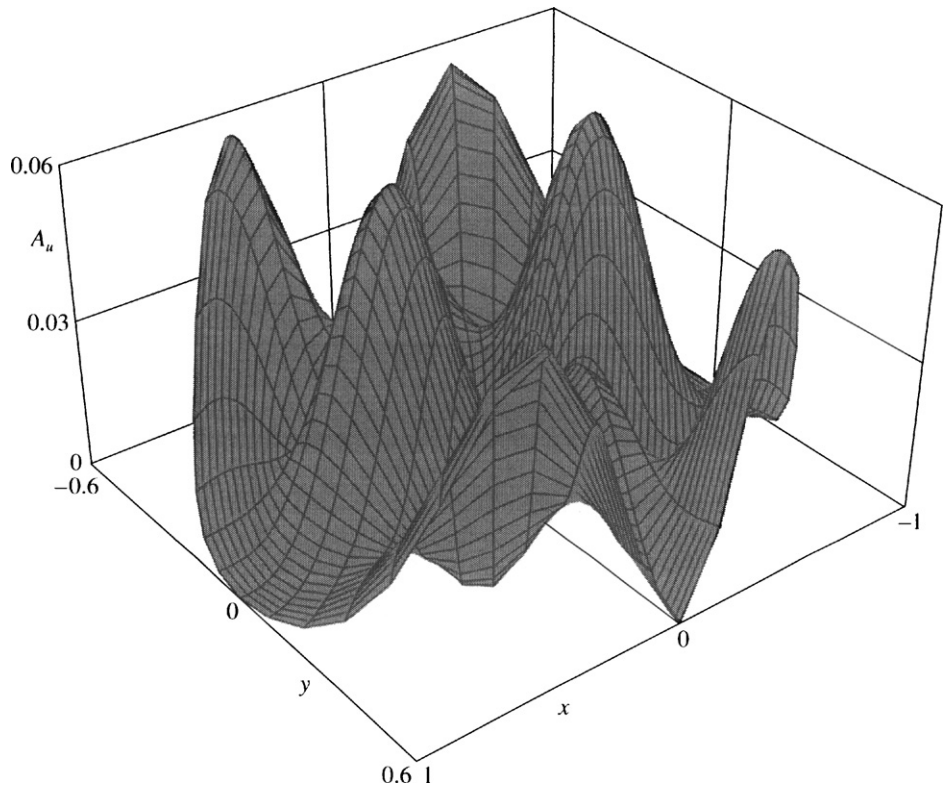

Fig. 4.

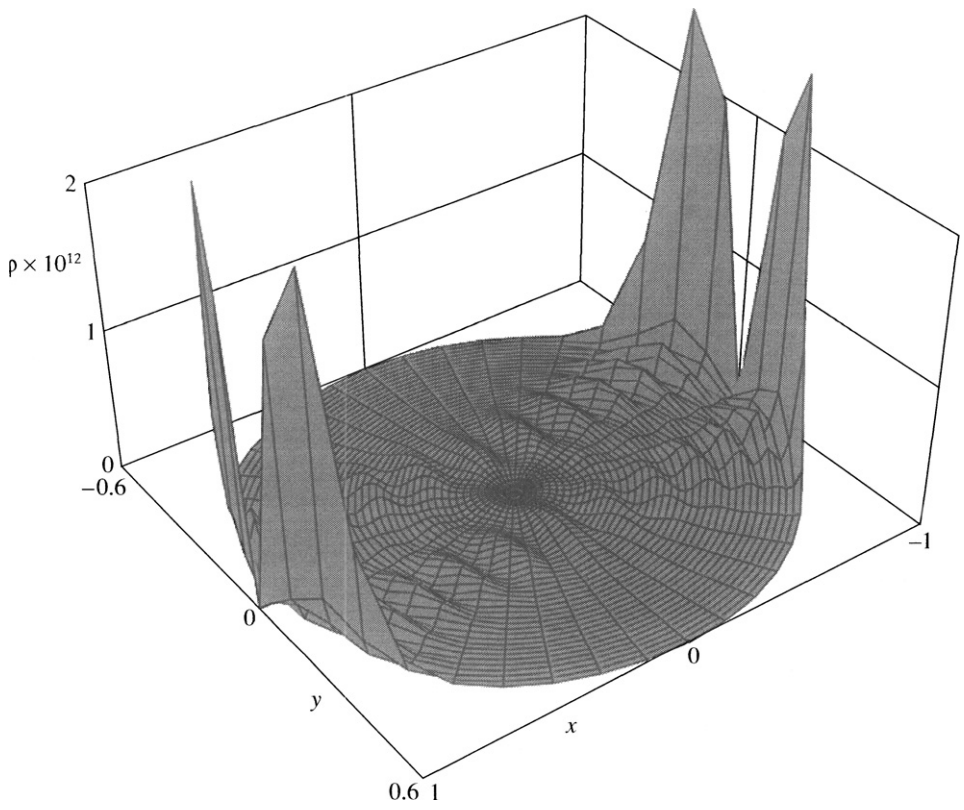

Fig. 5.

Table 3

Type of oscillations	Reλ	Imλ
III	3.490	5.2×10^{-12}
II	8.905	1.3×10^{-11}
I	11.466	9.4×10^{-9}
IV	16.864	7.1×10^{-8}
III	23.571	6.6×10^{-7}
II	28.273	1.3×10^{-6}
I	32.690	4.2×10^{-6}
I	39.545	3.2×10^{-4}
IV	43.267	3.5×10^{-4}
III	47.218	4.2×10^{-5}

Acknowledgements

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